MATH 579: Combinatorics
Exam 1 Solutions

1. How many four-digit numbers are there, with all their digits different? Note that 0123 is not a four-digit number since it leads with zero.

   We successively pick the digits from left to right. There are 9 choices for the leading digit (anything but zero). Regardless of our choice, there are 9 choices for the next digit (anything but the first digit). Regardless of our choices so far, there are 8 choices for the third digit (anything but the first two digits). There are similarly 7 choices for the last digit (anything but the first three digits). Hence the answer is $9 \cdot 9 \cdot 8 \cdot 7 = 4536$.

2. An "lc-word" is a string of letters, drawn from the 26 lowercase letters. How many four-letter lc-words are there, containing at least one vowel?

   Ignoring the vowel condition, lc-words of length n may be modeled as lists of length n, drawn from [26]. Therefore, there are 26^4 four-letter lc-words. From this we will subtract the number of lc-words containing no vowels at all. These latter may be modeled as lists of length 4, drawn from [21] (the set of consonants), of which there are 21^4. Hence the answer is $26^4 - 21^4 = 456,976 - 194,481 = 262,495$.

3. How many five-letter lc-words are there, containing exactly four distinct letters?

   The condition is equivalent to having one letter appear twice, and three letters appear once each. There are several procedures one may use to generate such words, one such follows. First, let's pick the letter appearing twice: 26 choices. Then, let's pick the two positions for this letter: $\binom{5}{2} = 10$ choices. At this point there are three empty positions, which we fill with three distinct letters, chosen from the 25 remaining. There are $25^{\underline{3}} = 13800$ choices. Putting it all together, we get $26 \cdot 10 \cdot 25^{\underline{3}} = 3,588,000$.

4. How many five-digit numbers are there, containing exactly four distinct digits?

   SOLUTION 1 (by cases): We split the desired numbers into two cases: First, those where the leading digit is repeated; second, those where two other digits are repeated. First case: there are 9 choices for the leading digit (not 0), and $\binom{4}{1} = 4$ choices for the location of the repetition. There are now three empty positions, which we fill with three distinct digits, chosen from the 9 remaining. There are $9^{\underline{3}} = 504$ choices. Combining, we get $9 \cdot 4 \cdot 9^{\underline{3}} = 18144$ numbers of the first case. Second case: there are again 9 choices for the leading digit, and $\binom{4}{2} = 6$ choices for the location of the two repeated digits. There are 9 choices for that repeated digit, and $8^{\underline{2}} = 56$ choices for the two remaining positions. Combining, we get $9 \cdot 6 \cdot 9 \cdot 8^{\underline{2}} = 27216$ numbers in the second case. Together, the answer is $18144 + 27216 = 45360$.

   SOLUTION 2: Avoiding cases requires a delicate and careful construction procedure. First, we choose which two digits will repeat; there are $\binom{5}{2} = 10$ choices. We will go from left to right (leading to trailing). Whichever the second position is for the repeated digit, we will cross out (as it will have been determined already). This leaves four positions unknown. The first (leading) digit has 9 choices (not 0). The next three digits have $9^{\underline{3}} = 504$ choices, as they can be modeled as an lode drawn from [9] (not

the leading digit). The second position is predetermined, as described already. Hence there are $9 \cdot 9^{\underline{3}} = 4536$ numbers with the prescribed repeated digits, and therefore $10 \cdot 4536 = 45360$ desired numbers.

5. How many five-digit numbers are there, whose digits alternate between even and odd? (e.g. 12345)

There are two cases, depending on whether there are three odd digits (and two even), or two odd digits (and three even). For the first case, OEOEO, there are $5^3 = 125$ ways to specify the three odd digits, as they are each independently chosen from $\{1, 3, 5, 7, 9\}$, and $5^2 = 25$ ways to specify the even digits, which are each independently chosen from $\{0, 2, 4, 6, 8\}$. Hence, there are $5^3 \cdot 5^2 = 3125$ such numbers. The second case, EOEOE, is more subtle, as the leading coefficient is even but cannot be zero. Hence, there are $4 \cdot 5^2 = 100$ ways to specify the even digits, and $5^2 = 25$ ways to specify the odd digits. Hence, there are $4 \cdot 5^2 \cdot 5^2 = 2500$ such numbers. Putting this together, the answer is $3125 + 2500 = 5625$.

6. How many five-card hands (from a standard 52-card deck) contain two different pairs, and a fifth unmatched card? In poker, this is called "two pair".

We initially distinguish the pairs as first pair vs. second pair. There are $\binom{13}{1} = 13$ ways to choose the rank for the first pair, and $\binom{4}{2} = 6$ ways to choose the suits for the first pair. Hence there are $13 \cdot 6 = 78$ ways to choose the first pair. There are $\binom{12}{1} = 12$ ways to choose the rank for the second pair (it can't be the same rank as the first pair), and $\binom{4}{2} = 6$ ways to choose the suits for the second pair. Hence there are $12 \cdot 6 = 72$ ways to choose the second pair. Our fifth card must not be of the denomination of either pair, so there are $\binom{11}{1} = 11$ denominations to choose from, and 4 suits, leaving $4 \cdot 11 = 44$ choices. We combine this to get $78 \cdot 72 \cdot 44 = 247104$. However, this overcounts what we are trying to count, as the "first pair" vs. "second pair" distinction is artificial and not part of what we're counting. Hence, each of the hands we want to count appears twice ($= 2!$) among our 247104 items, once for each of the possible orderings of the two pairs. Hence we are overcounting exactly twice, and the desired answer is $\frac{247104}{2} = 123552$.